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Non-Linear Diffusion II. Constitutive Equations for Mixtures of Isotropic Fluids

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NON-LINEAR DIFFUSION

II. CONSTITUTIVE EQUATIONS FOR MIXTURES OF ISOTROPIC FLUIDS

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CONTENTS

In the preceding paper I (Adkins 1963) a theory for the diffusion and flow of fluid mixtures was formulated based upon hydrodynamical considerations. It was assumed that the mechanical properties of each constituent of the mixture could be described by means of constitutive equations for the stresses occurring in the equations of motion for that constituent, while the effect of the other components could be accounted for by the nature of the body forces in these equations.

In the present work, some of the restrictions previously imposed upon the constitutive equations for the stresses and the body forces are removed. It is assumed that the stresses for a given constituent may depend upon velocity and acceleration gradients for all components of the mixture and that the body forces depend upon relative velocities and accelerations and upon velocity gradients. Invariance requirements are discussed, attention being confined to mixtures of isotropic fluids.

1. Introduction

In the preceding paper (part I) (Adkins 1963), a non-linear theory for the diffusion and flow of mixtures of fluids has been formulated based upon purely hydrodynamical considerations. The approach is essentially that proposed by Truesdell & Toupin (1960) and developed further, for linear systems, by Truesdell (1961). Each point of a mixture of n substances is assumed to be occupied simultaneously by the n constituents in given proportions. For each substance we may then define, at a given point, mechanical and kinematic quantities such as density, velocity, acceleration, a stress tensor and a body force vector. The mechanical properties of each individual component are described by constitutive equations for the stresses. Part of the body force acting on a given constituent is assumed to arise from the diffusion process and is given by constitutive equations describing the composition of the mixture and its motion.

In part I it has been assumed that the stresses for a given component $\mathscr{S}_r$ depend only upon the density of $\mathscr{S}_r$ and upon kinematic quantities referring explicitly to that substance; the diffusive forces were assumed to depend only on the composition of the mixture at a given point and the relative velocities of its constituents.

In the present paper some of these restrictions are removed. In § 3 the situation is examined in which the stresses for a given constituent depend upon the velocity gradients for all components of the mixture. In § 4, rotationally independent time derivatives of vectors and tensors are considered as a preliminary to an examination of the case in which there is dependence of the stresses upon mixed acceleration gradients. The final sections are concerned with constitutive equations for the body force vectors when mixed acceleration components and velocity gradients are taken into account. It would be possible to assume that the stresses also depend upon relative velocities and accelerations and the modification which this entails is indicated in § 7.

Owing to the rapidly increasing complexity of the analysis, attention is confined to situations in which only velocities and accelerations and their gradients are involved in the constitutive equations, dependence upon higher-order time derivatives being excluded. Furthermore, attention is concentrated throughout upon mixtures of isotropic materials. The analysis follows the lines indicated by Green & Rivlin (1960) and by Green & Adkins (1960) for single component systems; an alternative approach to the invariance problem for such systems has been given by Coleman & Noll (1959).

2. Notation and formulae

We consider a mixture of n substances $\mathscr{S}_r$ $(r = 1, 2, \ldots, n)$ which are in motion relative to each other, and assume that each point P within the mixture is occupied simultaneously by the substances $\mathscr{S}_r$, these being present in specified proportions.

We refer the motion to a fixed system of rectangular Cartesian co-ordinates x_i, and suppose that a particle of the substance $\mathscr{S}_r$ which is at y_i at the current time t was at the point $x_i^{(r)}$ at initial time $t = 0$. We assume that at the point y_i at time t the substance $\mathscr{S}_r$ has velocity $\mathbf{v}_r$ with components $v_i^{(r)}$ relative to the x_i axes. These components are given by

$$v_i^{(r)} = \frac{{}^{(r)}\mathrm{D}y_i}{\mathrm{D}t}, \quad y_i = y_i(x_k^{(r)}, t), \tag{2·1}$$

where ${}^{(r)}\mathrm{D}/\mathrm{D}t$ denotes differentiation with respect to t holding the co-ordinates $x_i^{(r)}$ constant.

If the density of the substance $\mathscr{S}_r$ at y_i is ρ_r, the density of the mixture is

$$\rho = \sum_{r=1}^{n} \rho_r, \tag{2·2}$$

and the mean velocity $\mathbf{v}$ of the mixture at this point is given by

$$\rho\mathbf{v} = \sum_{r=1}^{n} \rho_r \mathbf{v}_r. \tag{2·3}$$

The operator ${}^{(r)}\mathrm{D}/\mathrm{D}t$ is given by

$$\frac{{}^{(r)}\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + v_m^{(r)} \frac{\partial}{\partial y_m}, \tag{2·4}$$

where $\partial/\partial t$ denotes differentiation with respect to t holding the co-ordinates y_i constant; here and subsequently summation is carried out over repeated indices unless otherwise indicated. If we define the operator $\mathrm{D}/\mathrm{D}t$ by

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + v_m \frac{\partial}{\partial y_m}, \tag{2·5}$$

where v_m are the components of $\mathbf{v}$, then from (2·2) and (2·3)

$$\rho \frac{\mathrm{D}}{\mathrm{D}t} = \sum_{r=1}^{n} \rho_r \frac{{}^{(r)}\mathrm{D}}{\mathrm{D}t}. \tag{2·6}$$

We note that in operating on any quantity $\phi = \phi(t)$ which is independent of position, and therefore of y_i, the operators ${}^{(r)}\mathrm{D}/\mathrm{D}t$ and $\mathrm{D}/\mathrm{D}t$ reduce to $\partial/\partial t$.

We postulate that for each constituent $\mathscr{S}_r$ there exists at y_i a partial stress tensor $\boldsymbol{\sigma}_r$ with components $\sigma_{ij}^{(r)}$ and extraneous and diffusive body force vectors $\mathbf{F}_r$, $\boldsymbol{\Psi}_r$ with components $F_i^{(r)}$, $\Psi_i^{(r)}$ respectively referred to the x_i axes. The force $\boldsymbol{\Psi}_r$ acting on the substance $\mathscr{S}_r$ is assumed to arise from the influence of the other constituents of the mixture. For each constituent $\mathscr{S}_r$ we may then formulate equations of motion

$$\frac{\partial \sigma_{ik}^{(r)}}{\partial y_k} + \rho_r (F_i^{(r)} + \Psi_i^{(r)}) = \rho_r \frac{{}^{(r)}\mathrm{D} v_i^{(r)}}{\mathrm{D}t} \quad (r = 1, 2, \ldots, n;\ r \text{ not summed}), \tag{2·7}$$

and an equation of continuity

$$\frac{\partial \rho_r}{\partial t} + \frac{\partial}{\partial y_i}(\rho_r v_i^{(r)}) \equiv \frac{{}^{(r)}\mathrm{D}\rho_r}{\mathrm{D}t} + \rho_r \frac{\partial v_i^{(r)}}{\partial y_i} = 0 \quad (r = 1, 2, \ldots, n;\ r \text{ not summed}). \tag{2·8}$$

Equations (2·8) are modified if any of the substances $\mathscr{S}_r$ are removed or generated by chemical reactions or other processes; such cases are excluded from the present work.

It is assumed that diffusion effects can be accounted for by the nature of the body forces $\boldsymbol{\Psi}_r$, which therefore depend upon the composition of the mixture at the point P and upon the relative motions of its constituents. The properties of the individual constituents of the mixture are taken into account by the nature of the stress tensors $\boldsymbol{\sigma}_r$. In the previous work (part I), the assumption was made that the tensor $\boldsymbol{\sigma}_r$ depended only upon the density ρ_r and kinematic quantities defined for the substance $\mathscr{S}_r$; each of the body forces $\boldsymbol{\Psi}_r$ was assumed to depend upon the densities $\rho_1, \rho_2, \ldots, \rho_n$ and upon the relative velocities of the constituents $\mathscr{S}_r$. In subsequent sections we shall examine the consequences of more general assumptions.

The functional forms of $\boldsymbol{\Psi}_r$ and $\boldsymbol{\sigma}_r$ are restricted by the consideration that the properties of the mixture must be independent of rigid body motions of the medium as a whole. A simple technique for deriving the body force and stress components from suitably constructed scalar functions has been given by Pipkin & Rivlin (1960). If $\mathbf{p}$, $\mathbf{q}$ are arbitrary vectors with components p_i, q_i respectively at the point y_i, and we form the scalar functions

$$F^{(r)} = p_i \Psi_i^{(r)}, \quad G^{(r)} = p_i q_j \sigma_{ij}^{(r)}, \tag{2·9}$$

then $\Psi_i^{(r)}$ and $\sigma_{ij}^{(r)}$ are given uniquely by

$$\Psi_i^{(r)} = \frac{\partial F^{(r)}}{\partial p_i}, \quad \sigma_{ij}^{(r)} = \frac{\partial^2 G^{(r)}}{\partial p_i \partial q_j}. \tag{2·10}$$

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The invariance problem then reduces to that of constructing scalar functions $F^{(r)}$, $G^{(r)}$ of the appropriate kinematic and mechanical tensors which are linear in p_i and bilinear in p_i, q_j, respectively, are homogeneous in these quantities and which exhibit the required invariance properties.

In subsequent work we shall confine attention to the situation in which all components of the mixture are isotropic and the kinematic and mechanical tensors examined will be those appropriate to the isotropic case.

3. Dependence of stress upon velocity gradients

We examine first the situation which occurs when the stress components $\sigma_{ij}^{(r)}$ depend upon the velocity gradients $\partial v_i^{(s)}/\partial y_j$ $(s = 1, 2, \dots, n)$. Since the stresses can be derived from scalar functions $G^{(r)}$ which are homogeneous and bilinear in the components p_i, q_i by means of (2·10), it is sufficient to examine the form of the functions

$$G = G\left(\frac{\partial v_i^{(s)}}{\partial y_j}\right). \tag{3·1}$$

We assume that G is a polynomial in its arguments and since p_i, q_i are not involved in the subsequent discussion we do not exhibit these arguments explicitly.

We consider a motion $\mathscr{M}_r$ of the body which differs from the actual motion $\mathscr{M}$ defined by (2·1) only to the extent of an arbitrary superposed rigid-body rotation. The co-ordinates $\bar{y}_i$ in the varied motion $\mathscr{M}_r$, of the particle P, referred to the fixed Cartesian system x_i, are given by

$$\bar{y}_i = M_{ik} y_k. \tag{3·2}$$

Here $M_{ik} = M_{ik}(t)$ are continuous functions of t satisfying the conditions

$$M_{il} M_{kl} = M_{li} M_{lk} = \delta_{ik}, \quad |M_{ik}| = 1, \tag{3·3}$$

δ_{ik} being the Kronecker delta.

The velocity components $\bar{v}_i^{(r)}$ for the motion $\mathscr{M}_r$ are given by

$$\bar{v}_i^{(r)} = {}^{(r)}\mathrm{D}\bar{y}_i/\mathrm{D}t, \quad \bar{y}_i = \bar{y}_i(x_k^{(r)}, t). \tag{3·4}$$

From (2·1), (3·2) and (3·4) we then have

$$\bar{v}_i^{(r)} = M_{ij} v_j^{(r)} + y_j \mathrm{D}M_{ij}/\mathrm{D}t, \tag{3·5}$$

if we remember that M_{ij} are independent of the co-ordinates y_i.

From (3·3) we obtain

$$M_{jm}\frac{\mathrm{D}M_{im}}{\mathrm{D}t} = -M_{im}\frac{\mathrm{D}M_{jm}}{\mathrm{D}t} = \alpha_{ij} \quad \text{(say)}, \tag{3·6}$$

and by making use of this relation and (3·2) in (3·5) we find that

$$\bar{v}_i^{(r)} = M_{ij} v_j^{(r)} + \alpha_{ij}\bar{y}_j. \tag{3·7}$$

From (3·6) it follows that α_{ij} is skew symmetric, and if we define the vector $\Omega_k = \Omega_k(t)$ by the equations

$$\alpha_{ij} = \epsilon_{jik}\Omega_k, \tag{3·8}$$

ϵ_{jik} being the alternating tensor, we see that Ω_k are the components of the superposed rigid body motion referred to the x_i axes.

From (3·7) and (3·2) we have

$$\frac{\partial \bar{v}_i^{(r)}}{\partial \bar{y}_j} = M_{ik} M_{jl} \frac{\partial v_k^{(r)}}{\partial y_l} + \alpha_{ij} \quad (r = 1, 2, \ldots, n). \tag{3·9}$$

Writing
$$A_{ik}^{(rs)} = \frac{1}{2}\left(\frac{\partial v_i^{(r)}}{\partial y_k} + \frac{\partial v_k^{(s)}}{\partial y_i}\right) = A_{ki}^{(sr)}, \quad \omega_{ik}^{(rs)} = \frac{1}{2}\left(\frac{\partial v_i^{(r)}}{\partial y_k} - \frac{\partial v_k^{(s)}}{\partial y_i}\right) = -\omega_{ki}^{(sr)}, \tag{3·10}$$

and denoting by $\bar{A}_{ik}^{(rs)}$, $\bar{\omega}_{ik}^{(rs)}$ the corresponding quantities with $v_i^{(r)}$, $v_k^{(s)}$, y replaced by $\bar{v}_i^{(r)}$, $\bar{v}_k^{(s)}$, $\bar{y}$ respectively, we obtain from (3·9)

$$\left.\begin{aligned} \bar{A}_{ij}^{(rs)} &= M_{ik} M_{jl} A_{kl}^{(rs)}, \\ \bar{\omega}_{ij}^{(rs)} &= M_{ik} M_{jl} \omega_{kl}^{(rs)} + \alpha_{ij}. \end{aligned}\right\} \tag{3·11}$$

Hence
$$\bar{\omega}_{ij}^{(rs)} - \bar{\omega}_{ij}^{(pq)} = M_{ik} M_{jl} (\omega_{kl}^{(rs)} - \omega_{kl}^{(pq)}), \tag{3·12}$$

where p, q, r, s may take any of the values $1, 2, \ldots, n$. If at the instant t the two configurations represented by (2·1) and (3·2) coincide, $M_{ik} = \delta_{ik}$ and

$$\left.\begin{aligned} \bar{A}_{ij}^{(rs)} = A_{ij}^{(rs)}, \quad \bar{\omega}_{ij}^{(rs)} &= \omega_{ij}^{(rs)} + \alpha_{ij}, \\ \bar{\omega}_{ij}^{(rs)} - \bar{\omega}_{ij}^{(pq)} &= \omega_{ij}^{(rs)} - \omega_{ij}^{(pq)}. \end{aligned}\right\} \tag{3·13}$$

Returning to (3·1), we observe that since, from (3·10)

$$\frac{\partial v_i^{(r)}}{\partial y_j} = A_{ij}^{(rs)} + \omega_{ij}^{(rs)}, \tag{3·14}$$

the function G may be expressed as a polynomial in the components $A_{ij}^{(rs)}$ and $\omega_{ij}^{(rs)}$. Alternatively we may write G in the polynomial form

$$G = G(A_{ij}^{(rs)}, \omega_{ij}^{(rs)} - \omega_{ij}^{(nn)}, \omega_{ij}^{(nn)}). \tag{3·15}$$

From (3·13) the arguments $A_{ij}^{(rs)}$, $\omega_{ij}^{(rs)} - \omega_{ij}^{(nn)}$ are independent of the angular velocity α_{ij} of the superposed rigid body rotation in the motion $\mathscr{M}_r$, but $\omega_{ij}^{(nn)}$ depend upon α_{ij}. If, therefore, G is to be independent of the angular velocity of the superposed rigid body motion, the arguments $\omega_{ij}^{(nn)}$ must be excluded and we may write

$$G = G(A_{ij}^{(rs)}, \omega_{ij}^{(rs)} - \omega_{ij}^{(nn)}). \tag{3·16}$$

This form may be simplified if we observe from (3·10) that

$$\left.\begin{aligned} A_{ij}^{(rs)} &= \tfrac{1}{2}[A_{ij}^{(rr)} + A_{ij}^{(ss)} + \Omega_{ij}^{(r)} - \Omega_{ij}^{(s)}], \\ \omega_{ij}^{(rs)} - \omega_{ij}^{(nn)} &= \tfrac{1}{2}[A_{ij}^{(rr)} - A_{ij}^{(ss)} + \Omega_{ij}^{(r)} + \Omega_{ij}^{(s)}], \end{aligned}\right\} \tag{3·17}$$

where
$$\Omega_{ij}^{(r)} = \omega_{ij}^{(rr)} - \omega_{ij}^{(nn)}. \tag{3·18}$$

We may therefore replace (3·16) by

$$G = G(A_{ij}^{(rr)}, \Omega_{ij}^{(s)}) \quad (r = 1, 2, \ldots, n;\ s = 1, 2, \ldots, n-1). \tag{3·19}$$

The functional forms of G in (3·1), (3·16) and (3·19) are, in general, different.

From (3·19) we see that G may be expressed as a polynomial in the $6n$ components $A_{ij}^{(rr)}$ of the rate of deformation tensors and the $3n-3$ components $\Omega_{kl}^{(s)}$ expressing the rate of rotation of $n-1$ constituents relative to the remaining constituent $\mathscr{S}_n$.

As an alternative to the form (3·19) we may express G in terms of the gradients of the relative velocities of the constituents of the mixture. If we write

$$V_{ij}^{(rs)} = \frac{\partial}{\partial y_j}(v_i^{(r)} - v_i^{(s)}) \quad (r \neq s), \tag{3·20}$$

we obtain from (3·9)
$$\bar{V}_{ij}^{(rs)} = M_{ik} M_{jl} V_{kl}^{(rs)}, \tag{3·21}$$

and from (3·20) and (3·10)

$$\frac{\partial v_i^{(r)}}{\partial y_j} = V_{ij}^{(rn)} + A_{ij}^{(nn)} + \omega_{ij}^{(nn)} \quad (r = 1, 2, \ldots, n-1). \tag{3·22}$$

The polynomial function (3·1) may therefore be reduced to the form

$$G = G(V_{ij}^{(rn)}, A_{ij}^{(nn)}, \omega_{ij}^{(nn)}), \tag{3·23}$$

and in (3·23) only the arguments $\omega_{ij}^{(nn)}$ depend upon the angular velocity specified by α_{ij}. These must therefore be excluded, and we have

$$G = G(V_{ij}^{(rn)}, A_{ij}^{(nn)}) \quad (r = 1, 2, \ldots, n-1). \tag{3·24}$$

By the foregoing analysis, the $9n$ arguments $\partial v_i^{(s)}/\partial y_j$ occurring in the form (3·1) for G have been reduced, in (3·19) to the $9n-3$ components $A_{ij}^{(rr)}$, $\Omega_{ij}^{(s)}$, and in (3·24) to the $9n-3$ components $V_{ij}^{(rn)}$, $A_{ij}^{(nn)}$. This is consistent with the observation that the $9n$ arguments in (3·1) involve the three parameters α_{ij} defining the angular velocity of a rigid body rotation. Elimination of these must, in general, reduce by three the number of independent variables appearing in the form for G. From (3·10) and (3·20) we see that for a general deformation, the arguments appearing in (3·19) or in (3·24) are mutually independent and we may conclude that these forms contain no redundant elements.

Restoring the components p_i, q_i, we may infer that when the stresses $\sigma_{ij}^{(r)}$ are dependent upon the velocity gradients $\partial v_i^{(s)}/\partial y_j$, the functions in (2·10) may be written in the alternative forms

$$G^{(r)} = G^{(r)}(p_i, q_i;\ A_{ij}^{(tt)}, \Omega_{ij}^{(s)}), \tag{3·25}$$

$$G^{(r)} = G^{(r)}(p_i, q_i;\ V_{ij}^{(sn)}, A_{ij}^{(nn)}) \quad (r, t = 1, 2, \ldots, n;\ s = 1, 2, \ldots, n-1), \tag{3·26}$$

corresponding to (3·19) and (3·24), respectively. The functions $G^{(r)}$ in (3·25) and (3·26) are polynomials in their arguments and homogeneous and bilinear in p_i, q_i.

The forms (3·25) and (3·26) for $G^{(r)}$ are those appropriate to isotropic materials. This follows from the fact that in the motion $\mathscr{M}_r$, which incorporates the rigid body motion described by (3·2) and (3·3), the arguments occurring in (3·25) and (3·26) are replaced by the barred quantities defined by

$$\bar{A}_{ij}^{(tt)} = M_{ik} M_{jl} A_{kl}^{(tt)}, \quad \bar{\Omega}_{ij}^{(s)} = M_{ik} M_{jl} \Omega_{kl}^{(s)},$$
$$\bar{V}_{ij}^{(sn)} = M_{ik} M_{jl} V_{kl}^{(sn)},$$
$$\bar{p}_i = M_{ik} p_k, \quad \bar{q}_i = M_{ik} q_k.$$

If therefore the forms of $G^{(r)}$ are to be independent of rigid body displacements, that is, of M_{ik}, we must have

$$G^{(r)} = G^{(r)}(\bar{p}_i, \bar{q}_j;\ \bar{A}_{ij}^{(tt)}, \bar{\Omega}_{ij}^{(s)}) = G^{(r)}(p_i, q_j;\ A_{ij}^{(tt)}, \Omega_{ij}^{(s)}),$$

$$G^{(r)} = G^{(r)}(\bar{p}_i, \bar{q}_j;\ \bar{V}_{ij}^{(sn)}, \bar{A}_{ij}^{(nn)}) = G^{(r)}(p_i, q_j;\ V_{ij}^{(sn)}, A_{ij}^{(nn)}) \quad (r, t = 1, 2, \ldots, n;\ s = 1, 2, \ldots, n-1),$$

and these relations imply that in each case $G^{(r)}$ is an isotropic function of its arguments. The functions (3·25), (3·26) may be expressed as polynomials in the scalar invariants formed from the appropriate system of vectors and tensors. An integrity basis for the system of vectors and symmetric and skew symmetric tensors occurring in (3·25) may be deduced from the work of Spencer & Rivlin (to be published).

4. Rotationally invariant time derivatives of tensors

As a preliminary to the incorporation of time derivatives of relative velocities, diffusive force, velocity gradients and stress into the constitutive equations we construct forms for the time derivatives of tensors and vectors which are independent of superposed rigid body motions. The analysis follows the lines of that given by Green & Rivlin (1960) and Green & Adkins (1960) for single component systems, but in the present instance the existence of the n velocities $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ for the n components of the mixture needs to be taken into account.

We consider first a second-order tensor whose components at time t during the motions $\mathscr{M}$, $\mathscr{M}_r$ specified by (2·1) and (3·2) are

$$\psi_{ij} = \psi_{ij}(y_k, t), \quad \bar{\psi}_{ij} = \psi_{ij}(\bar{y}_k, t), \tag{4·1}$$

respectively, with

$$\bar{\psi}_{ij} = M_{ik} M_{jl} \psi_{kl}, \tag{4·2}$$

the quantities $M_{ik} = M_{ik}(t)$ satisfying (3·3). If the two configurations of the mixture at the instant t under consideration coincide, then $M_{ik} = \delta_{ik}$ and $\bar{\psi}_{ij} = \psi_{ij}$.

The time derivatives ${}^{(r)}\mathrm{D}\psi_{ij}/\mathrm{D}t$ are not independent of rigid body rotations, for, from (4·2) we have

$$\frac{{}^{(r)}\mathrm{D}\bar{\psi}_{ij}}{\mathrm{D}t} = M_{ik} M_{jl} \frac{{}^{(r)}\mathrm{D}\psi_{kl}}{\mathrm{D}t} + \left(M_{ik} \frac{\mathrm{D}M_{jl}}{\mathrm{D}t} + M_{jl} \frac{\mathrm{D}M_{ik}}{\mathrm{D}t} \right) \psi_{kl}. \tag{4·3}$$

However, with the help of (3·6), (3·9) and (4·2) we may obtain

$$\left. \begin{aligned} \bar{\psi}_{ij}^{(rst)} &= M_{ik} M_{jl} \psi_{kl}^{(rst)}, \\ \bar{\psi}_{ij}^{*(rst)} &= M_{ik} M_{jl} \psi_{kl}^{*(rst)}, \end{aligned} \right\} \tag{4·4}$$

where

$$\left. \begin{aligned} \psi_{ij}^{(rst)} &= \frac{{}^{(r)}\mathrm{D}\psi_{ij}}{\mathrm{D}t} + \psi_{im} \frac{\partial v_m^{(s)}}{\partial y_j} + \psi_{mj} \frac{\partial v_m^{(t)}}{\partial y_i}, \\ \psi_{ij}^{*(rst)} &= \frac{{}^{(r)}\mathrm{D}\psi_{ij}}{\mathrm{D}t} - \psi_{im} \frac{\partial v_j^{(s)}}{\partial y_m} - \psi_{mj} \frac{\partial v_i^{(t)}}{\partial y_m}, \end{aligned} \right\} \tag{4·5}$$

and r, s, t may take any of the integral values 1 to n inclusive.

From (4·4) we see that $\psi_{ij}^{(rst)}$, $\psi_{ij}^{*(rst)}$ are independent of the angular velocity of superposed rigid body motions. This is evidently true for any linear combination of these quantities, and in particular for the tensor defined by

$$\begin{aligned} \psi_{ij}^{\prime(rst)} &= \tfrac{1}{2}(\psi_{ij}^{(rst)} + \psi_{ij}^{*(rst)}) \\ &= {}^{(r)}\mathrm{D}\psi_{ij}/\mathrm{D}t + \psi_{im}\omega_{mj}^{(ss)} + \psi_{mj}\omega_{mi}^{(tt)}, \end{aligned} \tag{4·6}$$

the second form following from (4·5) with the help of (3·10).

A polynomial involving ψ_{ij} and ${}^{(r)}\mathrm{D}\psi_{ij}/\mathrm{D}t$ may therefore be replaced by one involving ψ_{ij}, $\partial v_i^{(r)}/\partial y_m$ and one of the systems of elements $\psi_{ij}^{(rst)}$, $\psi_{ij}^{*(rst)}$ or $\psi_{ij}^{\prime(rst)}$. Not all of the latter are required, however, for we may observe that

$$\left.\begin{aligned} \psi_{ij}^{(rst)}-\psi_{ij}^{(rpq)} &= \psi_{im}V_{mj}^{(sp)}+\psi_{mj}V_{mi}^{(tq)}, \\ \psi_{ij}^{*(rst)}-\psi_{ij}^{*(rpq)} &= -(\psi_{im}V_{jm}^{(sp)}+\psi_{mj}V_{im}^{(tq)}), \\ \psi_{ij}^{\prime(rst)}-\psi_{ij}^{\prime(rpq)} &= \psi_{im}(\omega_{mj}^{(ss)}-\omega_{mj}^{(pp)})+\psi_{mj}(\omega_{mi}^{(tt)}-\omega_{mi}^{(qq)}). \end{aligned}\right\} \tag{4·7}$$

Furthermore

$$\psi_{ij}^{(rst)} = {}^{(rn)}\mathrm{D}\psi_{ij}/\mathrm{D}t+\psi_{ij}^{(nst)}, \tag{4·8}$$

where, from (2·4)

$$\frac{{}^{(rs)}\mathrm{D}\psi_{ij}}{\mathrm{D}t} = \frac{{}^{(r)}\mathrm{D}\psi_{ij}}{\mathrm{D}t}-\frac{{}^{(s)}\mathrm{D}\psi_{ij}}{\mathrm{D}t} = (v_m^{(r)}-v_m^{(s)})\frac{\partial\psi_{ij}}{\partial y_m}, \tag{4·9}$$

and from (4·3)

$$\frac{{}^{(rs)}\mathrm{D}\bar{\psi}_{ij}}{\mathrm{D}t} = M_{ik}M_{jl}\frac{{}^{(rs)}\mathrm{D}\psi_{kl}}{\mathrm{D}t}. \tag{4·10}$$

Corresponding relations hold for $\psi_{ij}^{*(rst)}$ and $\psi_{ij}^{\prime(rst)}$.

From (4·7) and (4·8) we may write

$$\left.\begin{aligned} \psi_{ij}^{(rst)} &= \frac{{}^{(rn)}\mathrm{D}\psi_{ij}}{\mathrm{D}t}+{}^{(1)}\psi_{ij}+\psi_{im}V_{mj}^{(sn)}+\psi_{mj}V_{mi}^{(tn)}, \\ \psi_{ij}^{*(rst)} &= \frac{{}^{(rn)}\mathrm{D}\psi_{ij}}{\mathrm{D}t}+{}^{(1)}\psi_{ij}^{*}-\psi_{im}V_{jm}^{(sn)}-\psi_{mj}V_{im}^{(tn)}, \\ \psi_{ij}^{\prime(rst)} &= \frac{{}^{(rn)}\mathrm{D}\psi_{ij}}{\mathrm{D}t}+\frac{{}^{(n)}\mathscr{D}\psi_{ij}}{\mathscr{D}t}+\psi_{im}\Omega_{mj}^{(s)}+\psi_{mj}\Omega_{mi}^{(t)}, \end{aligned}\right\} \tag{4·11}$$

where

$$\left.\begin{aligned} {}^{(1)}\psi_{ij} \equiv \psi_{ij}^{(nnn)} &= \frac{{}^{(n)}\mathrm{D}\psi_{ij}}{\mathrm{D}t}+\psi_{im}\frac{\partial v_m^{(n)}}{\partial y_j}+\psi_{mj}\frac{\partial v_m^{(n)}}{\partial y_i} && \text{(i)}, \\ {}^{(1)}\psi_{ij}^{*} \equiv \psi_{ij}^{*(nnn)} &= \frac{{}^{(n)}\mathrm{D}\psi_{ij}}{\mathrm{D}t}-\psi_{im}\frac{\partial v_j^{(n)}}{\partial y_m}-\psi_{mj}\frac{\partial v_i^{(n)}}{\partial y_m} && \text{(ii)}, \\ \frac{{}^{(n)}\mathscr{D}\psi_{ij}}{\mathscr{D}t} &= \tfrac{1}{2}({}^{(1)}\psi_{ij}+{}^{(1)}\psi_{ij}^{*}) \\ &= \frac{{}^{(n)}\mathrm{D}\psi_{ij}}{\mathrm{D}t}+\psi_{im}\omega_{mj}^{(nn)}+\psi_{mj}\omega_{mi}^{(nn)} && \text{(iii)}. \end{aligned}\right\} \tag{4·12}$$

Higher-order time derivatives of the tensor ψ_{ij} may be incorporated into the analysis in a similar manner. For example, corresponding to the first of (4·11) we may form the derivative

$$\begin{aligned} \psi_{ij}^{(r_1s_1t_1,r_2s_2t_2)} = {} & \frac{{}^{(r_1n)}\mathrm{D}}{\mathrm{D}t}\left[\frac{{}^{(r_2n)}\mathrm{D}\psi_{ij}}{\mathrm{D}t}\right]+\frac{{}^{(r_1n)}\mathrm{D}[{}^{(1)}\psi_{ij}]}{\mathrm{D}t}+{}^{(1)}\left[\frac{{}^{(r_2n)}\mathrm{D}\psi_{ij}}{\mathrm{D}t}\right] \\ & +{}^{(2)}\psi_{ij}+\frac{{}^{(r_1n)}\mathrm{D}}{\mathrm{D}t}\{\psi_{im}V_{mj}^{(s_2n)}+\psi_{mj}V_{mi}^{(t_2n)}\}+{}^{(1)}[\psi_{im}V_{mj}^{(s_2n)}+\psi_{mj}V_{mi}^{(t_2n)}] \\ & +\psi_{im}^{(r_2s_2t_2)}V_{mj}^{(s_1n)}+\psi_{mj}^{(r_2s_2t_2)}V_{mi}^{(t_1n)}, \end{aligned} \tag{4·13}$$

where ${}^{(2)}\psi_{ij}$ is obtained by replacing ψ_{ij} by ${}^{(1)}\psi_{ij}$ in (4·12) (i) and ${}^{(1)}[\ \ ]$ denotes the expression obtained from (4·12) (i) by replacing ψ_{ij} by the terms in square brackets. Corresponding forms may be derived based upon $\psi_{ij}^{*(rst)}$ and $\psi_{ij}^{\prime(rst)}$.

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From (4·11) we observe that the system of derivatives ${}^{(r)}\mathrm{D}\psi_{ij}/\mathrm{D}t$ may be replaced by a system involving the $n-1$ derivatives ${}^{(rn)}\mathrm{D}\psi_{ij}/\mathrm{D}t$ together with one of the derivatives ${}^{(1)}\psi_{ij}$, ${}^{(1)}\psi^{*}_{ij}$ or ${}^{(n)}\mathscr{D}\psi_{ij}/\mathscr{D}t$ and terms not involving derivatives of ψ_{ij}. Similarly, in (4·13) the system of n^2 second derivatives $[{}^{(r_1)}\mathrm{D}/\mathrm{D}t]\,[{}^{(r_2)}\mathrm{D}\psi_{ij}/\mathrm{D}t]$ $(r_1, r_2 = 1, 2, \ldots, n)$ is replaced by a system of n^2 second derivatives exemplified by the first four terms on the right-hand side of this equation together with terms involving lower-order derivatives of ψ_{ij}. In (4·13), operators of the form ${}^{(r)}\mathrm{D}/\mathrm{D}t$, ${}^{(s)}\mathrm{D}/\mathrm{D}t$ for $r \neq s$ do not, in general, commute, for if ϕ is any differentiable function of y_i and t

$$\left[\frac{{}^{(r)}\mathrm{D}}{\mathrm{D}t}\frac{{}^{(s)}\mathrm{D}}{\mathrm{D}t}-\frac{{}^{(s)}\mathrm{D}}{\mathrm{D}t}\frac{{}^{(r)}\mathrm{D}}{\mathrm{D}t}\right]\phi = \left[\left(v^{(r)}_m\frac{\partial v^{(s)}_l}{\partial y_m}-v^{(s)}_m\frac{\partial v^{(r)}_l}{\partial y_m}\right)-\frac{\partial}{\partial t}\left(v^{(r)}_l-v^{(s)}_l\right)\right]\frac{\partial\phi}{\partial y_l}. \tag{4·14}$$

A corresponding analysis may be carried out for a tensor of any rank. We may, in particular, write down the results for a vector whose components at time t for the motions $\mathscr{M}$, $\mathscr{M}_r$ specified by (2·1) and (3·2) are

$$v_i = v_i(y_k, t), \quad \bar{v}_i = v_i(\bar{y}_k, t), \tag{4·15}$$

respectively, with

$$\bar{v}_i = M_{ik}v_k. \tag{4·16}$$

In this case, the first derivatives independent of rigid body rotations are

$$v^{(rs)}_i = \frac{{}^{(r)}\mathrm{D}v_i}{\mathrm{D}t}+v_m\frac{\partial v^{(s)}_m}{\partial y_i} = \frac{{}^{(rn)}\mathrm{D}v_i}{\mathrm{D}t}+{}^{(1)}v_i+v_mV^{(sn)}_{mi}, \tag{4·17}$$

$$v^{*(rs)}_i = \frac{{}^{(r)}\mathrm{D}v_i}{\mathrm{D}t}-v_m\frac{\partial v^{(s)}_i}{\partial y_m} = \frac{{}^{(rn)}\mathrm{D}v_i}{\mathrm{D}t}+{}^{(1)}v^{*}_i-v_mV^{(sn)}_{im}, \tag{4·18}$$

$$v'^{(rs)}_i = \tfrac{1}{2}(v^{(rs)}_i+v^{*(rs)}_i) = \frac{{}^{(rn)}\mathrm{D}v_i}{\mathrm{D}t}+\frac{{}^{(n)}\mathscr{D}v_i}{\mathscr{D}t}+v_m\Omega^{(s)}_{mi}, \tag{4·19}$$

where

$$\left.\begin{aligned} {}^{(1)}v_i \equiv v^{(nn)}_i &= \frac{{}^{(n)}\mathrm{D}v_i}{\mathrm{D}t}+v_m\frac{\partial v^{(n)}_m}{\partial y_i},\\ {}^{(1)}v^{*}_i \equiv v^{*(nn)}_i &= \frac{{}^{(n)}\mathrm{D}v_i}{\mathrm{D}t}-v_m\frac{\partial v^{(n)}_i}{\partial y_m},\\ \frac{{}^{(n)}\mathscr{D}v_i}{\mathscr{D}t} = \tfrac{1}{2}({}^{(1)}v_i+{}^{(1)}v^{*}_i) &= \frac{{}^{(n)}\mathrm{D}v_i}{\mathrm{D}t}+v_m\omega^{(nn)}_{mi}. \end{aligned}\right\} \tag{4·20}$$

Each of the tensors $\psi^{(rst)}_{ij}$, $\psi^{*(rst)}_{ij}$, $\psi'^{(rst)}_{ij}$, ${}^{(1)}\psi_{ij}$, ${}^{(1)}\psi^{*}_{ij}$, ${}^{(n)}\mathscr{D}\psi_{ij}/\mathscr{D}t$, $\psi^{(r_1s_1t_1,r_2s_2t_2)}_{ij}$ defined in (4·11), (4·12) and (4·13) satisfies a transformation law of the form (4·2) and this remains valid if ψ_{ij} is replaced by one of the kinematic tensors $A^{(rs)}_{ij}$, $\omega^{(rs)}_{ij}-\omega^{(pq)}_{ij}$, $V^{(rs)}_{ij}$ or $\Omega^{(r)}_{ij}$ defined in § 3, or by the stress tensor $\sigma^{(r)}_{ij}$. Similarly, each of the vectors $v^{(rs)}_i$, $v^{*(rs)}_i$, $v'^{(rs)}_i$, ${}^{(1)}v_i$, ${}^{(1)}v^{*}_i$, ${}^{(n)}\mathscr{D}v_i/\mathscr{D}t$ defined in (4·17) to (4·20) satisfies a transformation law of the form (4·16), and this remains true if v_i is replaced by one of the relative velocities $v^{(r)}_i-v^{(s)}_i$ or by the body force components $\Psi^{(r)}_i$.

5. Dependence of stress upon velocity and acceleration gradients

From (2·1) we may define successive acceleration components for the substance $\mathscr{S}_r$ at P by the relations

$${}^{(2)}v^{(r)}_i = \frac{{}^{(r)}\mathrm{D}v^{(r)}_i}{\mathrm{D}t}, \quad \ldots, \quad {}^{(p)}v^{(r)}_i = \frac{{}^{(r)}\mathrm{D}^{(p-1)}v^{(r)}_i}{\mathrm{D}t}. \tag{5·1}$$

More generally, we may introduce mixed time derivatives of the form

$$\left.\begin{aligned} v_i^{(sr)} &= \frac{{}^{(s)}\mathrm{D}v_i^{(r)}}{\mathrm{D}t}, \quad v_i^{(tsr)} = \frac{{}^{(t)}\mathrm{D}v_i^{(sr)}}{\mathrm{D}t}, \\ v^{(r_q r_{q-1} \dots r_1)} &= \frac{{}^{(r_q)}\mathrm{D}}{\mathrm{D}t}\frac{{}^{(r_{q-1})}\mathrm{D}}{\mathrm{D}t}, \quad \dots, \quad \frac{{}^{(r_1)}\mathrm{D}y_i}{\mathrm{D}t}, \end{aligned}\right\} \tag{5·2}$$

where $r, s, t, r_q, \dots$ may take any of the values $1, 2, \dots, n$. From these expressions we may form the gradients

$$\partial v_i^{(sr)}/\partial y_j,\ \partial v_i^{(tsr)}/\partial y_j,\ \dots.$$

For simplicity, we restrict attention to the case in which the stresses $\sigma_{ij}^{(t)}$ depend upon the velocity gradients $\partial v_i^{(r)}/\partial y_j$ and the acceleration gradients $\partial v_i^{(rs)}/\partial y_j$ and for this purpose it is sufficient to examine the scalar function

$$G = G\left(\frac{\partial v_i^{(r)}}{\partial y_j}, \frac{\partial v_i^{(st)}}{\partial y_j}\right) \quad (r, s, t = 1, 2, \dots, n), \tag{5·3}$$

where G is a polynomial in the arguments indicated.

From (5·2), (2·4) and (3·10) we have

$$\left.\begin{aligned} \frac{{}^{(p)}\mathrm{D}A_{ij}^{(rs)}}{\mathrm{D}t} &= \frac{1}{2}\left[\frac{\partial v_i^{(pr)}}{\partial y_j} + \frac{\partial v_j^{(ps)}}{\partial y_i} - \frac{\partial v_m^{(p)}}{\partial y_j}\frac{\partial v_i^{(r)}}{\partial y_m} - \frac{\partial v_m^{(p)}}{\partial y_i}\frac{\partial v_j^{(s)}}{\partial y_m}\right], \\ \frac{{}^{(p)}\mathrm{D}\omega_{ij}^{(rs)}}{\mathrm{D}t} &= \frac{1}{2}\left[\frac{\partial v_i^{(pr)}}{\partial y_j} - \frac{\partial v_j^{(ps)}}{\partial y_i} - \frac{\partial v_m^{(p)}}{\partial y_j}\frac{\partial v_i^{(r)}}{\partial y_m} + \frac{\partial v_m^{(p)}}{\partial y_i}\frac{\partial v_j^{(s)}}{\partial y_m}\right], \end{aligned}\right\} \tag{5·4}$$

and hence, remembering (3·18)

$$\begin{aligned} \frac{\partial v_i^{(pr)}}{\partial y_j} &= \frac{{}^{(p)}\mathrm{D}A_{ij}^{(rr)}}{\mathrm{D}t} + \frac{{}^{(p)}\mathrm{D}\omega_{ij}^{(rr)}}{\mathrm{D}t} + \frac{\partial v_m^{(p)}}{\partial y_j}\frac{\partial v_i^{(r)}}{\partial y_m} \\ &= \frac{{}^{(p)}\mathrm{D}A_{ij}^{(rr)}}{\mathrm{D}t} + \frac{{}^{(p)}\mathrm{D}\Omega_{ij}^{(r)}}{\mathrm{D}t} + \frac{{}^{(p)}\mathrm{D}\omega_{ij}^{(nn)}}{\mathrm{D}t} + \frac{\partial v_m^{(p)}}{\partial y_j}\frac{\partial v_i^{(r)}}{\partial y_m}. \end{aligned} \tag{5·5}$$

The functional form (5·3) may therefore be replaced by

$$G = G\left(\frac{\partial v_i^{(r)}}{\partial y_j}, \frac{{}^{(p)}\mathrm{D}A_{ij}^{(ss)}}{\mathrm{D}t}, \frac{{}^{(p)}\mathrm{D}\Omega_{ij}^{(tt)}}{\mathrm{D}t}, \frac{{}^{(p)}\mathrm{D}\omega_{ij}^{(nn)}}{\mathrm{D}t}\right) \quad (p, r, s = 1, 2, \dots, n;\ t = 1, 2, \dots, n-1), \tag{5·6}$$

the polynomial structure of G being preserved.

From (4·9) and (4·12), for the derivatives of tensor ψ_{ij} we have the alternative expressions

$$\begin{aligned} \frac{{}^{(p)}\mathrm{D}\psi_{ij}}{\mathrm{D}t} &= \frac{{}^{(pn)}\mathrm{D}\psi_{ij}}{\mathrm{D}t} + {}^{(1)}\psi_{ij} - \psi_{im}\frac{\partial v_m^{(n)}}{\partial y_j} - \psi_{mj}\frac{\partial v_m^{(n)}}{\partial y_i} \\ &= \frac{{}^{(pn)}\mathrm{D}\psi_{ij}}{\mathrm{D}t} + {}^{(1)}\psi_{ij}^{*} + \psi_{im}\frac{\partial v_j^{(n)}}{\partial y_m} + \psi_{mj}\frac{\partial v_i^{(n)}}{\partial y_m} \\ &= \frac{{}^{(pn)}\mathrm{D}\psi_{ij}}{\mathrm{D}t} + \frac{{}^{(n)}\mathscr{D}\psi_{ij}}{\mathscr{D}t} - \psi_{im}\omega_{mj}^{(nn)} - \psi_{mj}\omega_{mi}^{(nn)}, \end{aligned} \tag{5·7}$$

and in these formulae ψ_{ij} may be replaced by $A_{ij}^{(rs)}$ or $\Omega_{ij}^{(r)}$. With the aid of (3·10), (4·9) and (5·7) we infer that the form (5·6) may be reduced to one of the forms

$$\left.\begin{aligned} G &= G\left(\frac{\partial v_i^{(r)}}{\partial y_j}, \frac{^{(pn)}\mathrm{D}A_{ij}^{(rr)}}{\mathrm{D}t}, \frac{^{(pn)}\mathrm{D}\Omega_{ij}^{(s)}}{\mathrm{D}t}, {}^{(1)}A_{ij}^{(rr)}, {}^{(1)}\Omega_{ij}^{(s)}, \frac{^{(pn)}\mathrm{D}\omega_{ij}^{(nn)}}{\mathrm{D}t}, \frac{^{(n)}\mathrm{D}\omega_{ij}^{(nn)}}{\mathrm{D}t}\right),\\ G &= G\left(\frac{\partial v_i^{(r)}}{\partial y_j}, \frac{^{(pn)}\mathrm{D}A_{ij}^{(rr)}}{\mathrm{D}t}, \frac{^{(pn)}\mathrm{D}\Omega_{ij}^{(s)}}{\mathrm{D}t}, {}^{(1)}A_{ij}^{*(rr)}, {}^{(1)}\Omega_{ij}^{*(s)}, \frac{^{(pn)}\mathrm{D}\omega_{ij}^{(nn)}}{\mathrm{D}t}, \frac{^{(n)}\mathrm{D}\omega_{ij}^{(nn)}}{\mathrm{D}t}\right),\\ G &= G\left(\frac{\partial v_i^{(r)}}{\partial y_j}, \frac{^{(pn)}\mathrm{D}A_{ij}^{(rr)}}{\mathrm{D}t}, \frac{^{(pn)}\mathrm{D}\Omega_{ij}^{(s)}}{\mathrm{D}t}, \frac{^{(n)}\mathscr{D}A_{ij}^{(rr)}}{\mathscr{D}t}, \frac{^{(n)}\mathscr{D}\Omega_{ij}^{(s)}}{\mathscr{D}t}, \frac{^{(pn)}\mathrm{D}\omega_{ij}^{(nn)}}{\mathrm{D}t}, \frac{^{(n)}\mathrm{D}\omega_{ij}^{(nn)}}{\mathrm{D}t}\right)\\ &\qquad (r = 1, 2, \ldots, n;\ p, s = 1, 2, \ldots, n-1), \end{aligned}\right\} \tag{5·8}$$

and if we make use of the reductions of § 3, these become

$$\left.\begin{aligned} G &= G\left(A_{ij}^{(rr)}, \Omega_{ij}^{(s)}, \frac{^{(pn)}\mathrm{D}A_{ij}^{(rr)}}{\mathrm{D}t}, \frac{^{(pn)}\mathrm{D}\Omega_{ij}^{(s)}}{\mathrm{D}t}, {}^{(1)}A_{ij}^{(rr)}, {}^{(1)}\Omega_{ij}^{(s)}, \frac{^{(pn)}\mathrm{D}\omega_{ij}^{(nn)}}{\mathrm{D}t}, \frac{^{(n)}\mathrm{D}\omega_{ij}^{(nn)}}{\mathrm{D}t}, \omega_{ij}^{(nn)}\right),\\ G &= G\left(A_{ij}^{(rr)}, \Omega_{ij}^{(s)}, \frac{^{(pn)}\mathrm{D}A_{ij}^{(rr)}}{\mathrm{D}t}, \frac{^{(pn)}\mathrm{D}\Omega_{ij}^{(s)}}{\mathrm{D}t}, {}^{(1)}A_{ij}^{*(rr)}, {}^{(1)}\Omega_{ij}^{*(s)}, \frac{^{(pn)}\mathrm{D}\omega_{ij}^{(nn)}}{\mathrm{D}t}, \frac{^{(n)}\mathrm{D}\omega_{ij}^{(nn)}}{\mathrm{D}t}, \omega_{ij}^{(nn)}\right),\\ G &= G\left(A_{ij}^{(rr)}, \Omega_{ij}^{(s)}, \frac{^{(pn)}\mathrm{D}A_{ij}^{(rr)}}{\mathrm{D}t}, \frac{^{(pn)}\mathrm{D}\Omega_{ij}^{(s)}}{\mathrm{D}t}, \frac{^{(n)}\mathscr{D}A_{ij}^{(rr)}}{\mathscr{D}t}, \frac{^{(n)}\mathscr{D}\Omega_{ij}^{(s)}}{\mathscr{D}t}, \frac{^{(pn)}\mathrm{D}\omega_{ij}^{(nn)}}{\mathrm{D}t}, \frac{^{(n)}\mathrm{D}\omega_{ij}^{(nn)}}{\mathrm{D}t}, \omega_{ij}^{(nn)}\right)\\ &\qquad (r = 1, 2, \ldots, n;\ p, s = 1, 2, \ldots, n-1), \end{aligned}\right\} \tag{5·9}$$

respectively.

From (3·11), by differentiation we have

$$\frac{^{(p)}\mathrm{D}\overline{\omega}_{ij}^{(rs)}}{\mathrm{D}t} = M_{ik}M_{jl}\frac{^{(p)}\mathrm{D}\omega_{kl}^{(rs)}}{\mathrm{D}t} + \left(M_{ik}\frac{\mathrm{D}M_{jl}}{\mathrm{D}t} + M_{jl}\frac{\mathrm{D}M_{ik}}{\mathrm{D}t}\right)\omega_{kl}^{(rs)} + \frac{\mathrm{D}\alpha_{ij}}{\mathrm{D}t}, \tag{5·10}$$

and therefore
$$\frac{^{(pn)}\mathrm{D}\overline{\omega}_{ij}^{(rs)}}{\mathrm{D}t} = M_{ik}M_{jl}\frac{^{(pn)}\mathrm{D}\omega_{kl}^{(rs)}}{\mathrm{D}t}. \tag{5·11}$$

In the motion $\mathscr{M}_r$, the quantities $^{(p)}\mathrm{D}\omega_{ij}^{(nn)}/\mathrm{D}t$ thus depend upon the angular acceleration $\mathrm{D}\alpha_{ij}/\mathrm{D}t$ of the superposed rigid body motion and the quantities $\omega_{ij}^{(nn)}$ depend upon the angular velocity of this motion. They must therefore be omitted from the functional forms (5·9) if G is to be independent of superposed rigid body motions. From (5·11), however, we see that the quantities $^{(pn)}\mathrm{D}\omega_{ij}^{(nn)}/\mathrm{D}t$ may be retained. The function (5·3) may therefore be reduced to one of the forms

$$\left.\begin{aligned} G &= G\left(A_{ij}^{(rr)}, \Omega_{ij}^{(s)}, \frac{^{(pn)}\mathrm{D}A_{ij}^{(rr)}}{\mathrm{D}t}, \frac{^{(pn)}\mathrm{D}\Omega_{ij}^{(s)}}{\mathrm{D}t}, {}^{(1)}A_{ij}^{(rr)}, {}^{(1)}\Omega_{ij}^{(s)}, \frac{^{(pn)}\mathrm{D}\omega_{ij}^{(nn)}}{\mathrm{D}t}\right),\\ G &= G\left(A_{ij}^{(rr)}, \Omega_{ij}^{(s)}, \frac{^{(pn)}\mathrm{D}A_{ij}^{(rr)}}{\mathrm{D}t}, \frac{^{(pn)}\mathrm{D}\Omega_{ij}^{(s)}}{\mathrm{D}t}, {}^{(1)}A_{ij}^{*(rr)}, {}^{(1)}\Omega_{ij}^{*(s)}, \frac{^{(pn)}\mathrm{D}\omega_{ij}^{(nn)}}{\mathrm{D}t}\right),\\ G &= G\left(A_{ij}^{(rr)}, \Omega_{ij}^{(s)}, \frac{^{(pn)}\mathrm{D}A_{ij}^{(rr)}}{\mathrm{D}t}, \frac{^{(pn)}\mathrm{D}\Omega_{ij}^{(s)}}{\mathrm{D}t}, \frac{^{(n)}\mathscr{D}A_{ij}^{(rr)}}{\mathscr{D}t}, \frac{^{(n)}\mathscr{D}\Omega_{ij}^{(s)}}{\mathscr{D}t}, \frac{^{(pn)}\mathrm{D}\omega_{ij}^{(nn)}}{\mathrm{D}t}\right)\\ &\qquad (r = 1, 2, \ldots, n;\ p, s = 1, 2, \ldots, n-1). \end{aligned}\right\} \tag{5·12}$$

In (5·8) to (5·12) $^{(1)}A_{ij}^{(rr)}$, $^{(1)}\Omega_{ij}^{(s)}$ are obtained by replacing ψ_{ij} by $A_{ij}^{(rr)}$, $\Omega_{ij}^{(s)}$ respectively in the first of (4·12); $^{(1)}A_{ij}^{*(rr)}$, $^{(1)}\Omega_{ij}^{*(s)}$, $^{(n)}\mathscr{D}A_{ij}^{(rr)}/\mathscr{D}t$, $^{(n)}\mathscr{D}\Omega_{ij}^{(s)}/\mathscr{D}t$ are derived similarly from the second and third of (4·12).

Forms for G analogous to (3·23) may be derived by making use of the relations

$$\left.\begin{aligned} 2A_{ij}^{(rr)} &= V_{ij}^{(rn)} + V_{ji}^{(rn)} + 2A_{ij}^{(nn)}, \\ 2\Omega_{ij}^{(r)} &= V_{ij}^{(rn)} - V_{ji}^{(rn)}, \end{aligned}\right\} \tag{5·13}$$

which follow from (3·20). We may then replace the system of arguments $A_{ij}^{(rr)}$, $\Omega_{ij}^{(r)}$ by $V_{ij}^{(rn)}$, $A_{ij}^{(nn)}$ in (5·12). For example, the first of (5·12) yields

$$G = G\left(V_{ij}^{(rn)},\ A_{ij}^{(nn)},\ \frac{{}^{(pn)}\mathrm{D}V_{ij}^{(rn)}}{\mathrm{D}t},\ \frac{{}^{(pn)}\mathrm{D}A_{ij}^{(nn)}}{\mathrm{D}t},\ {}^{(1)}V_{ij}^{(rn)},\ {}^{(1)}A_{ij}^{(nn)},\ \frac{{}^{(pn)}\mathrm{D}\omega_{ij}^{(nn)}}{\mathrm{D}t}\right) \quad (p, r = 1, 2, \ldots, n-1), \tag{5·14}$$

and corresponding expressions may be obtained from the second and third of (5·12).

It follows from the argument of § 3 that (5·12) and (5·14) must be isotropic functions of their arguments. As before, explicit expressions for the stresses may be obtained by restoring the components p_i, q_i to the forms in (5·12) and (5·14) and making use of (2·10).

6. Dependence of diffusive force upon velocity and acceleration components

In the earlier work attention has been confined to the situation in which the diffusive forces $\mathbf{\Psi}_r$ depend upon the composition of the mixture at the point y_i and upon the relative velocities of its constituents. We extend consideration here to the case in which accelerations enter into the constitutive equations and examine the scalar functions

$$F = F(v_i^{(r)}, v_i^{(st)}) \quad (v_i^{(st)} = {}^{(s)}\mathrm{D}v_i^{(t)}/\mathrm{D}t). \tag{6·1}$$

In (6·1), F is a polynomial in the arguments indicated which depends also upon the densities ρ_r. The latter are scalar quantities which do not affect the invariance problem and are not exhibited explicitly.

If F is to be independent of rigid body translations the arguments in (6·1) must enter as differences

$$v_i^{(r)} - v_i^{(s)}, \quad v_i^{(pq)} - v_i^{(rs)}. \tag{6·2}$$

These are invariant under the transformation

$$\bar{y}_i = y_i + \phi_i(t), \tag{6·3}$$

where $\phi_i(t)$ are functions of t independent of position, specifying a rigid body translational motion. It is sufficient to write F in the functional form

$$F = F(u_i^{(r)}, v_i^{(rs)} - v_i^{(nn)}), \tag{6·4}$$

where

$$u_i^{(r)} = v_i^{(r)} - v_i^{(n)}. \tag{6·5}$$

This reduces by six the number of independent arguments in F consistent with the reduction needed to eliminate the six components of velocity and acceleration $\partial\phi_i/\partial t$, $\partial^2\phi_i/\partial t^2$ of the rigid body translation (6·3).

From (3·5) we see that

$$\bar{v}_i^{(rs)} = M_{ik} v_k^{(rs)} + \frac{\mathrm{D}M_{ik}}{\mathrm{D}t}(v_k^{(r)} + v_k^{(s)}) + \frac{\mathrm{D}^2 M_{ik}}{\mathrm{D}t^2} y_k. \tag{6·6}$$

It follows that if

$$\left.\begin{aligned} a_i^{(rs)} &= \frac{{}^{(rn)}\mathrm{D}u_i^{(s)}}{\mathrm{D}t} = v_i^{(rs)} - v_i^{(rn)} - v_i^{(ns)} + v_i^{(nn)}, \\ b_i^{(rs)} &= v_i^{(rs)} - v_i^{(sr)} = -b_i^{(sr)}, \end{aligned}\right\} \tag{6·7}$$

then $$\bar{a}_i^{(rs)} = M_{ik} a_k^{(rs)}, \quad \bar{b}_i^{(rs)} = M_{ik} b_k^{(rs)}, \tag{6·8}$$

and since, from (6·7)

$$v_i^{(ns)} - v_i^{(nn)} = v_i^{(sn)} - v_i^{(nn)} - b_i^{(sn)},$$

$$v_i^{(rs)} - v_i^{(nn)} = a_i^{(rs)} - b_i^{(sn)} + (v_i^{(rn)} - v_i^{(nn)}) + (v_i^{(sn)} - v_i^{(nn)}),$$

the function F may be reduced to the form

$$F = F(u_i^{(r)}, a_i^{(rs)}, b_i^{(rs)}, v_i^{(rn)} - v_i^{(nn)}) \quad (r, s = 1, 2, \ldots, n-1). \tag{6·9}$$

In (6·9) the differences $v_i^{(rn)} - v_i^{(nn)}$ depend upon the angular velocity of the rigid body motion in $\mathscr{M}_r$; this is also true of any linear combination of these differences provided the velocities $\mathbf{v}_r$ are linearly independent. Non-linear terms containing $v_i^{(rn)} - v_i^{(nn)}$ which are independent of the additional motion may, however, be constructed. For example, if we write

$$B^{(rs)} = u_i^{(r)}(v_i^{(sn)} - v_i^{(nn)}) + u_i^{(s)}(v_i^{(rn)} - v_i^{(nn)})$$

$$= \frac{{}^{(n)}\mathrm{D}}{\mathrm{D}t}(u_i^{(r)} u_i^{(s)}) + b_i^{(rn)} u_i^{(s)} + b_i^{(sn)} u_i^{(r)}, \tag{6·10}$$

then since $\bar{u}_i^{(r)} \bar{u}_i^{(s)} = u_i^{(r)} u_i^{(s)}$ we have $\bar{B}^{(rs)} = B^{(rs)}$. The polynomial function

$$F = F(u_i^{(r)}, a_i^{(rs)}, b_i^{(rn)}, B^{(rs)}) \quad (r, s = 1, 2, \ldots, n-1) \tag{6·11}$$

is therefore a polynomial in the arguments contained in (6·4) or (6·9) which is independent of the angular velocity and acceleration of the rigid body motion contained in $\mathscr{M}_r$. For independence of rigid body displacements, F must be an isotropic function of its arguments.

7. Diffusive force dependent upon velocities, accelerations and velocity gradients

A more complete reduction can be achieved if we can assume that the diffusive forces $\mathbf{\Psi}_r$, or the function F, depends explicitly upon the velocity gradients $\partial v_i^{(s)}/\partial y_k$. In this case we write, in place of (6·1),

$$F = F\left(v_i^{(r)}, v_i^{(rs)}, \frac{\partial v_i^{(r)}}{\partial y_j}\right) \quad (r, s = 1, 2, \ldots, n), \tag{7·1}$$

and a reduction similar to that of § 6 leads to the form

$$F = F\left(u_i^{(r)}, a_i^{(rs)}, b_i^{(rn)}, v_i^{(rn)} - v_i^{(nn)}, \frac{\partial v_i^{(p)}}{\partial y_j}\right) \quad (r, s = 1, 2, \ldots, n-1;\ p = 1, 2, \ldots, n), \tag{7·2}$$

corresponding to (6·9).

From (6·7), (5·2) and (2·4) we now have

$$\left.\begin{aligned} a_i^{(rs)} &= u_m^{(r)} \partial u_i^{(s)}/\partial y_m \quad (u_i^{(s)} = v_i^{(s)} - v_i^{(n)}), \\ v_i^{(rn)} - v_i^{(nn)} &= u_m^{(r)} \partial v_i^{(n)}/\partial y_m, \end{aligned}\right\} \tag{7·3}$$

and (7·2) may be replaced by

$$F = F(u_i^{(r)}, b_i^{(rn)}, \partial v_i^{(p)}/\partial y_j) \quad (r = 1, 2, \ldots, n-1;\ p = 1, 2, \ldots, n). \tag{7·4}$$

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A reduction following the lines of § 3 for the function (3·1) leads to the permissible forms

$$\left.\begin{aligned} F &= F(u_i^{(r)}, b_i^{(rn)}, A_{ij}^{(pp)}, \Omega_{ij}^{(r)}),\\ F &= F(u_i^{(r)}, b_i^{(rn)}, V_{ij}^{(rn)}, A_{ij}^{(nn)}) \quad (r = 1, 2, \ldots, n-1;\ p = 1, 2, \ldots, n), \end{aligned}\right\} \tag{7·5}$$

for F corresponding to (3·19) and (3·24), respectively. As before, the functions (7·1), (7·2), (7·4) and (7·5) are (different) polynomials in the arguments indicated and for independence of rigid body displacements the forms (7·5) must be isotropic functions of their arguments.

An extension of the analysis to the case where the stresses $\sigma_{ik}^{(r)}$ are assumed to depend upon relative velocities and accelerations is obtained by including the arguments $u_i^{(r)}$, $b_i^{(rn)}$ in the forms for G derived in §§ 3 and 5.

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